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The textbook contains lecture material for the second part of the course on  
mathematical analysis and includes the following topics: indefinite integral, defi-  
nite inte-gral and its geometric applications, improper integral, numerical series,  
functional sequences and series, power series, Fourier series. A useful feature of  
the book is the possibility of studying the course material at the same time as  
viewing video lectures recorded by the author and available on youtube.com. Sec-  
tions and subsections of the textbook are provided with information about the  
lecture number, the start time of the corresponding fragment and the duration of  
this fragment. In the electronic version of the textbook, this information is pre-  
sented as hyperlinks, allowing reader to immediately view the required fragment  
of the lecture.

The textbook is intended for students specializing in science and engineering.

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